

THE GABRIEL-ROITER MEASURES ADMITTING NO DIRECT PREDECESSORS OVER n -KRONECKER QUIVERS

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ABSTRACT. Let Q be an n -Kronecker quiver, i.e., a quiver with two vertices, labeled by 1 and 2, and n arrows from 2 to 1. We show that there are infinitely many Gabriel-Roiter measures admitting no direct predecessors.

Keywords. Direct predecessor, Gabriel-Roiter measure, n -Kronecker modules.

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1. BACKGROUND, PRELIMINARIES AND THE MAIN RESULT

Let Λ be an artin algebra and $\text{mod } \Lambda$ the category of finitely generated right Λ -modules. For each $M \in \text{mod } \Lambda$, we denote by $|M|$ the length of M . The symbol \subset is used to denote proper inclusion.

We first recall the original definition of Gabriel-Roiter measure [11, 12]. Let $\mathbb{N}_1 = \{1, 2, \dots\}$ be the set of natural numbers and $\mathcal{P}(\mathbb{N}_1)$ be the set of all subsets of \mathbb{N}_1 . A total order on $\mathcal{P}(\mathbb{N}_1)$ can be defined as follows: if I, J are two different subsets of \mathbb{N}_1 , write $I < J$ if the smallest element in $(I \setminus J) \cup (J \setminus I)$ belongs to J . Also we write $I \ll J$ provided $I \subset J$ and for all elements $a \in I$, $b \in J \setminus I$, we have $a < b$. We say that J **starts with** I if $I = J$ or $I \ll J$. Thus $I < J < I'$ with I' starts with I implies that J starts with I .

For each $M \in \text{mod } \Lambda$, let $\mu(M)$ be the maximum of the sets $\{|M_1|, |M_2|, \dots, |M_t|\}$, where $M_1 \subset M_2 \subset \dots \subset M_t$ is a chain of indecomposable submodules of M . We call $\mu(M)$ the **Gabriel-Roiter (GR for short) measure** of M . A subset I of $\mathcal{P}(\mathbb{N}_1)$ is called a GR measure if there is an indecomposable module M with $\mu(M) = I$. If M is an indecomposable Λ -module, we call an inclusion $X \subset M$ with X indecomposable a **GR inclusion** provided $\mu(M) = \mu(X) \cup \{|M|\}$, thus if and only if every proper submodule of M has Gabriel-Roiter measure at most $\mu(X)$. In this case, we call X a **GR submodule** of M . Note that the factor of a GR inclusion is indecomposable.

Using Gabriel-Roiter measure, Ringel obtained a partition of the module category for any artin algebra of infinite representation type [11, 12]: there are infinitely many GR measures

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I_i and I^i with i natural numbers, such that

$$I_1 < I_2 < I_3 < \dots \quad \dots < I^3 < I^2 < I^1$$

and such that any other GR measure I satisfies $I_i < I < I^j$ for all i, j . The GR measures I_i (resp. I^i) are called take-off (resp. landing) measures. Any other GR measure is called a central measure. An indecomposable module M is called a take-off (resp. central, landing) module if its GR measure $\mu(M)$ is a take-off (resp. central, landing) measure.

Let I, I' be two GR measures for Λ . We call I' a **direct successor** of I if, first, $I < I'$ and second, there does not exist a GR measure I'' with $I < I'' < I'$. The so-called **Successor Lemma** in [12] states that any GR measure I different from I^1 , the maximal one, has a direct successor. However, there is no ‘Predecessor Lemma’. For example, the GR measure of a homogeneous simple module over a tame quiver of type \tilde{A}_n has no direct predecessor [5]. More general, for a tame quiver (of type $\tilde{A}_n, \tilde{D}_n, \tilde{E}_6, \tilde{E}_7$ or \tilde{E}_8), the minimal central GR measure exists, which obviously has no direct predecessor [4]. Let us denote by $\text{ndp}(\Lambda)$ the number of the GR measures admitting no direct predecessors. It is clear that any GR measure over a representation-finite artin algebra has a direct predecessor, i.e., $\text{ndp}(\Lambda) = 0$. It was shown in [5] that for each tame hereditary algebra over an algebraically closed field $\text{ndp}(\Lambda) < \infty$, and thus $1 \leq \text{ndp}(\Lambda) < \infty$ by above mentioned facts. Therefore, it is natural to ask if the number of the GR measures admitting no direct predecessors relates to representation type of hereditary algebras. More general, we want to know whether a finite dimensional algebra Λ over an algebraically closed field is of wild type implies that there are infinitely many GR measures admitting no direct predecessors, i.e., $\text{ndp}(\Lambda) = \infty$, and vice versa.

Let Q' be the following quiver:

$$1 \longrightarrow 2 \begin{array}{c} \rightrightarrows \\ \leftarrow \end{array} 3 ,$$

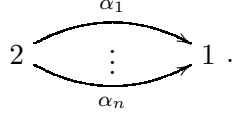
and $\text{mod}(Q)$ the category of finite dimensional representations over an algebraically closed field. In [6], it was proved that the GR measure $\mu^m = \{1, 2, 4, \dots, 2m, 2m+1\}$ has no direct predecessor for each $m \geq 1$. The embedding of the 2-Kronecker quiver into Q' was used in the discussion.

In this note, a second example of wild quiver with infinitely many GR measures admitting no direct predecessors will be presented. Namely the so called n -Kronecker quiver with $n \geq 3$ will be studied. Using the same method in [6], we show the following theorem:

Theorem. *Let Q be an n -Kronecker quiver with $n \geq 3$. Then for each $m \geq 1$, $\mu^m = \{1, 2, 4, \dots, 2m, 2m+1\}$ is a GR measure which does not admit a direct predecessor. Thus $\text{ndp}(Q) = \infty$.*

2. THE GR MEASURES FOR n -KRONECKER QUIVER

Let Q be the following n -Kronecker quiver:



Note that Q is of finite representation type if $n = 1$ and of tame type if $n = 2$. If $n \geq 3$, then Q is of wide representation type. We recall some facts of representations of quivers. The best references are [1, 10]. We also refer to [8, 9] for general structures of representations of wild quivers. Let k be an algebraically closed field. A representation for Q over k is simply called an n -Kronecker module.

The Cartan matrix and the Coxeter matrix are the following:

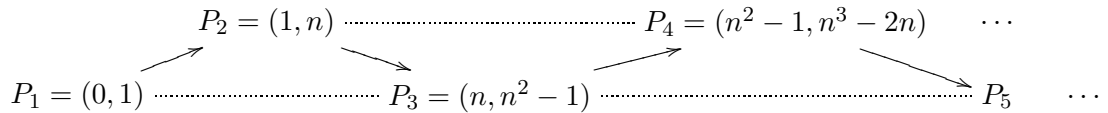
$$C = \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}, \quad \Phi = -C^{-t}C = \begin{pmatrix} n^2 - 1 & n \\ -n & -1 \end{pmatrix}, \quad \Phi^{-1} = \begin{pmatrix} -1 & -n \\ n & n^2 - 1 \end{pmatrix}$$

The dimension vectors can be calculated using $\underline{\dim} \tau M = (\underline{\dim} M)\Phi$ if M is not projective and $\underline{\dim} \tau^{-1}N = (\underline{\dim} N)\Phi^{-1}$ if N is not injective, where τ denotes the Auslander-Reiten translation. The Euler form is $\langle \underline{x}, \underline{y} \rangle = x_1y_1 + x_2y_2 - nx_1y_2$. Then for two indecomposable modules X and Y ,

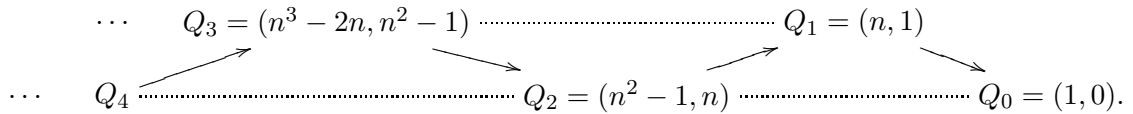
$$\dim \text{Hom}(X, Y) - \dim \text{Ext}^1(X, Y) = \langle \underline{\dim} X, \underline{\dim} Y \rangle.$$

Now let us assume that $n \geq 2$. The Auslander-Reiten quiver of Q consists of one preprojective component, one preinjective component and infinitely many regular ones. An indecomposable regular module X is called quasi-simple if the Auslander-Reiten sequence starting with X has an indecomposable middle term. For each indecomposable regular module M , there is a unique quasi-simple module X and a unique natural number $r \geq 1$ (called quasi-length of M and denoted by $\text{ql}(M) = r$) such that there is a sequence of irreducible monomorphisms $X = X[1] \rightarrow X[2] \rightarrow \dots \rightarrow X[r] = M$.

The preprojective component is the following (note that there are namely n arrows from P_i to P_{i+1}):



Similarly, the preinjective component is of the following shape:



2.1. The partition using GR measrue. Before describing the take-off part and the landing part of n -Kronecker quiver for $n \geq 2$. we need some properties of GR measures. The following property was proved in [11]:

Proposition 2.1. *Let Λ be an artin algebra and X and Y_1, Y_2, \dots, Y_r be indecomposable modules. Assume that $X \xrightarrow{f} \oplus_{i=1}^r Y_i$ is a monomorphism.*

- (1) $\mu(X) \leq \max\{\mu(Y_i)\}$.
- (2) If $\max\{\mu(Y_i)\} = \mu(X)$, then f splits.

We collect some properties of GR inclusions in the following lemma. The proof can be found for example in [2, 3].

Lemma 2.2. *Let Λ be an artin algebra and $X \subset M$ a GR inclusion.*

- (1) *If all irreducible maps to M are monomorphisms, then the GR inclusion is an irreducible map.*
- (2) *Every irreducible map to M/X is an epimorphism.*
- (3) *There is an irreducible monomorphism $X \rightarrow Y$ with Y indecomposable and an epimorphism $Y \rightarrow M$.*

Let us denote by I_i (resp. I^i) the take-off (resp. landing) measure obtained using GR measure and by $\mathcal{A}(I)$ the set of isomorphism classes of indecomposable modules (or representatives of these isomorphism classes) with GR measure I . Obviously, $I_1 = \{1\}$ and $\mathcal{A}(I_1)$ contains precisely all simple modules. It is easily seen that $I_2 = \{1, r\}$, where r is the largest possible length of a local module of Loewy length 2.

Proposition 2.3. *Let $n \geq 2$ and Q be the n -Kronecker quiver.*

- (1) $\mathcal{A}(I_r) = \{P_r\}$ for all $r \geq 2$. *Thus the take-off part contains precisely the simple injective module and the indecomposable preprojective modules.*
- (2) $\mathcal{A}(I^r) = \{Q_r\}$ for all $r \geq 1$. *Thus the landing part contains precisely the non-simple indecomposable preinjective module.*
- (3) *An indecomposable module is in central part if and only if it is a regular module.*

Proof. (1) By Lemma 2.2(1), P_i is the unique, up to isomorphism, GR submodule of P_{i+1} . We proceed by induction. For $r = 2$, the assertion holds by the description of I_2 . Assume that $\mu(M) = I_{r+1}$ for some indecomposable module M . Since M is not simple, we can assume that Y is a GR submodule of M . Then $\mu(Y) = I_i \leq I_r$ for some $i \leq r$, and thus $Y \cong P_i$ by induction. It follows from Lemma 2.2(3) that there is an epimorphism $P_{i+1} \rightarrow M$. In particular $|M| \leq |P_{i+1}|$. If the equality does not hold, then

$$I_{r+1} = \mu(M) = I_i \cup \{|M|\} > I_i \cup \{|P_{i+1}|, \dots, |P_r|, |P_{r+1}|\} > I_i \cup \{|P_{i+1}|, \dots, |P_r|\} = I_r.$$

This is a contradiction because the GR measure $\mu(P_{r+1}) = I_i \cup \{|P_{i+1}|, \dots, |P_r|, |P_{n+1}|\}$ lies between I_r and I_{r+1} , and I_{r+1} is a direct successor of I_r . Therefore, $|P_{i+1}| = |M|$ and thus $P_{i+1} \cong M$. Since $\mu(M) = I_{r+1}$, we have $i = n$ and thus $\mathcal{A}(I_{r+1}) = \{P_{r+1}\}$.

(2) It was proved in [11] that all modules lying in the landing part are preinjective (in more general sense). Since there is a short exact sequence $0 \rightarrow Q_{r+1} \rightarrow Q_r^n \rightarrow Q_{r-1} \rightarrow 0$ for each $r \geq 1$, $\mu(Q_{r+1}) < \mu(Q_r)$ by Proposition 2.1. Since landing modules are preinjective, $\mathcal{A}(I^r)$ contains precisely one isomorphism class, namely Q_r .

Statement (3) is a direct consequence of (1) and (2). \square

In general, the take-off part of a bimodule algebra can be similarly described. We refer to [13] for details.

2.2. The GR measures of 2-Kronecker modules. As an example, we describe the GR measures for 2-Kronecker modules to be used in our later discussion. These follow easily from direct calculation. For more properties of general tame quivers of type $\tilde{\mathbb{A}}_n$, we refer to [3, 5].

The GR measure of the indecomposable preprojective module with dimension vector $(m, m+1)$ is $\{1, 3, 5, \dots, 2m+1\}$.

Every indecomposable regular module with dimension vector (m, m) has GR measure $\{1, 2, 4, 6, \dots, 2m\}$.

The GR measure of the indecomposable preinjective module with dimension vector $(m+1, m)$ is $\{1, 2, 4, \dots, 2m, 2m+1\}$.

Remark. Comparing to the main theorem, the GR measure $\mu^m = \{1, 2, 4, \dots, 2m, 2m+1\}$ in 2-Kronecker case has a direct predecessor μ^{m+1} , for each $m \geq 1$.

2.3. GR measures admitting no direct predecessors. Now we assume that Q is of wild type, i.e., $n \geq 3$. The indecomposable regular modules X with dimension vector $(1, 1)$ or $(2, 1)$ are of special interest because they have no proper regular factor modules. Thus any non-zero homomorphism from X to an indecomposable regular M is a monomorphism.

Lemma 2.4. (1) *An indecomposable module M has GR measure $\{1, 2\}$ if and only if $\underline{\dim} M = (1, 1)$.*

(2) *An indecomposable module M has GR measure $\{1, 2, 3\}$ if and only if $\underline{\dim} M = (2, 1)$.*

Proof. Straightforward. \square

Lemma 2.5. *Let X be an indecomposable module with dimension vector $(1, 1)$. Then for each $i \geq 1$ the GR measure $\mu(\tau^i X)$ starts with $\{1, 2, 3\}$, i.e., $\mu(\tau^i X) = \{1, 2, 3, \dots\}$.*

Proof. If X is an indecomposable module with dimension vector $(1, 1)$. Then

$$\underline{\dim} \tau X = (1, 1) \begin{pmatrix} n^2 - 1 & n \\ -n & -1 \end{pmatrix} = (n^2 - n - 1, n - 1).$$

Thus $\dim \text{Hom}(X, \tau X) - \dim \text{Ext}^1(X, \tau X) = \langle \underline{\dim} X, \underline{\dim} \tau X \rangle = n^2 - n - 1 + n - 1 - n^2 + n = n - 2 \geq 1$ for each $n \geq 3$. In particular, $\text{Hom}(X, \tau X) \neq 0$ and there exists a monomorphism from X to τX . Therefore, there exists a monomorphism $\tau^i X \rightarrow \tau^{i+1} X$ for each $i \geq 0$. In particular, $\mu(\tau^i X) > \mu(\tau^j X)$ for $i > j \geq 1$.

Let Y be an indecomposable module with dimension vector $(2, 1)$. Then

$$\dim \text{Hom}(Y, \tau X) - \dim \text{Ext}^1(Y, \tau X) = \langle \underline{\dim} Y, \underline{\dim} \tau X \rangle = n - 3 \geq 0.$$

Thus if $n \geq 4$, then there is a monomorphism $Y \rightarrow \tau X$. In particular, $\mu(\tau^i X) \geq \mu(\tau X) > \mu(Y) = \{1, 2, 3\}$ for each $i \geq 1$.

Assume that $n = 3$. We show that τX contains an indecomposable submodule with dimension vector $(2, 1)$. Note that $\underline{\dim} \tau X = (5, 2)$. Since each indecomposable module with dimension vector $(2, 1)$ has no proper regular factor, it is sufficient to show that there is an indecomposable module Y with dimension vector $(2, 1)$ such that $\text{Hom}(Y, \tau X) \cong \mathbb{D} \text{Ext}^1(X, Y) \cong \text{Hom}(\tau^{-1} Y, X) \neq 0$. Thus it is sufficient to show that there is an indecomposable module Y' with dimension vector $(1, 2) = (2, 1)\Phi^{-1}$ such that $\text{Hom}(Y', X) \neq 0$. Let P_1 be the simple projective module. Then $\text{Ext}^1(X, S_1) \cong \mathbb{D} \text{Hom}(S_1, \tau X) \neq 0$. Thus there is a non-split short exact sequence $0 \rightarrow S_1 \rightarrow Y \rightarrow X \rightarrow 0$. It is easily seen that the middle term Y is indecomposable and has dimension vector $(1, 2)$. Since τX contains an indecomposable submodule with dimension vector $(2, 1)$, we have $\mu(\tau^i X) \geq \mu(\tau X) > \{1, 2, 3\}$ for all $i \geq 1$. \square

Lemma 2.6. *For each $m \geq 1$, let $\mu_m = \{1, 2, 4, \dots, 2m\}$.*

- (1) μ_m is a GR measure and if M is an indecomposable module with $\mu(M) = \mu_m$, then $\underline{\dim} M = (m, m)$.
- (2) If $\mu(M) = \mu_m$, then each indecomposable regular factor module of M contains some indecomposable submodule with dimension vector $(1, 1)$.

Proof. (1) Let M be an indecomposable 2-Kronecker module with dimension vector (m, m) . Then M is an indecomposable n -Kronecker module as well and M has GR measure μ_m (Section 2.2). Thus μ_m is a GR measure. Now assume that M is an indecomposable module with GR measure $\mu(M) = \mu_m$. If $m = 1$, then $\mu(M) = \{1, 2\}$ and thus $\underline{\dim} M = (1, 1)$. Assume that $m > 1$. A GR submodule N of M has GR measure μ_{m-1} . By induction, $\underline{\dim} N = (m-1, m-1)$. Since a GR factor of length 2 has dimension vector $(1, 1)$, $\underline{\dim} M = (m, m)$.

(2) We use induction on m . This is clear for $m = 1$. If $m = 2$, then the dimension vector of a proper indecomposable regular factor is $(1, 1)$ or $(2, 1)$. Now assume that $m \geq 2$. Let Y be a proper regular factor of M and K be the kernel of the canonical projective. Let N be a GR submodule of M . Then we have the following diagram:

$$\begin{array}{ccccccc} & & & N & & & \\ & & & \downarrow f & & & \\ 0 & \longrightarrow & K & \xrightarrow{\iota} & M & \xrightarrow{\pi} & Y \longrightarrow 0 \end{array}$$

If the composition πf is not zero, then the image of πf is a regular factor of N . Therefore by induction, $\text{im } \pi f$, and thus Y , contains a submodule with dimension vector $(1, 1)$. If the composition πf is zero, then l factors through ι by definition of kernel. In particular, there is a monomorphism $N \xrightarrow{g} K$. Since N is a GR submodule of M , it follows that N is isomorphic to a direct summand of K (Proposition 2.1). Since Y is not simple and $|K| \geq |X|$, we have $K \cong N$. Thus $|Y| = 2$ and $\underline{\dim} Y = (1, 1)$. \square

Lemma 2.7. *Let $\mu^m = \{1, 2, 4, \dots, 2m, 2m + 1\}$ for each $m \geq 1$. Then*

- (1) μ^m is a GR measure.
- (2) *If M is an indecomposable module with GR measure $\mu(M) = \mu^m$, then $\underline{\dim} M = (m + 1, m)$.*

Proof. It is known that each indecomposable 2-Kronecker module with dimension vector $(m + 1, m)$ has GR measure μ^m . Thus μ^m is a GR measure for n -Kronecker quiver. We have seen that an indecomposable module M has GR measure $\{1, 2, 4, \dots, 2m\}$ implies that $\underline{\dim} M = (m, m)$. Thus an indecomposable module with GR measure μ^m has dimension vector $(m + 1, m)$. \square

Corollary 2.8. *Let M be an indecomposable module with GR measure $\mu(M) = \mu^m$. Then each indecomposable regular factor module of M contains some indecomposable submodule with dimension vector $(1, 1)$.*

Proof. By Lemma 2.7, $\underline{\dim} M = (m + 1, m)$. we have a short exact sequence

$$0 \rightarrow N \xrightarrow{\iota} M \rightarrow M/N \rightarrow 0$$

where ι is a GR inclusion. Thus $\underline{\dim} N = (m, m)$ and the factor M/N is an injective simple module. Let $M \xrightarrow{\pi} Y$ be an epimorphism with Y an indecomposable regular module. Then $\text{Hom}(M/N, Y) = 0$. By definition of cokernel, the composition $N \xrightarrow{\pi \iota} Y$ is not zero. Therefore, by Lemma 2.6, the image of $\pi \iota$, and thus Y , contains a submodule with dimension vector $(1, 1)$. \square

Lemma 2.9. *Let $m \geq 1$ and M be an indecomposable module such that $\mu(M) > \mu^m$. Then $\mu(M)$ starts with $\mu(M) = \{1, 2, \dots, 2t, 2t + 1\}$ for some $1 \leq t \leq m$. In particular, M contains an indecomposable submodule with GR measure μ^t .*

Proof. This follows directly from the definition of GR measure. \square

Lemma 2.10. *If M is an indecomposable module such that $\mu = \mu(M)$ is a direct predecessor of μ^m for some m . Then M is regular and $|M| > 2m + 1$.*

Proof. Note that a direct predecessor of a central measure is a central measure as well. Thus M is regular since μ^m , and thus $\mu(M)$, is a central measure. Since $\mu(M)$ is a direct predecessor of μ^m , we have $\mu_m \leq \mu(M)$. However, the equality does not hold because $\mu_m < \mu_{m+1} < \mu^m$. Thus $\mu_m < \mu(M) < \mu^m$ and $|M| > 2m + 1$. \square

proof of Theorem. For the purpose of a contradiction, we assume that M is an indecomposable module such that $\mu(M)$ is a direct predecessor of μ^m for a fixed $m \geq 1$. Thus M is a regular module by Lemma 2.10 and $M \cong X[r]$ for some quasi-simple X and $r \geq 1$. Again by Lemma 2.10, we have $|X[r + 1]| > |X[r]| > 2m + 1$. It follows that $\mu(M) = \mu(X[r]) < \mu^m < \mu(X[r + 1])$. Thus $X[r + 1]$ contains a submodule Y with GR measure μ^t for some $1 \leq t \leq m$ (Lemma 2.9). Note that $\underline{\dim} Y = (t + 1, t)$ and $\mu(Y) \geq \mu^m$. We claim that $\text{Hom}(Y, \tau^{-r}X) = 0$. If this is not the case, then by Corollary 2.8, the image of a nonzero homomorphism, in particular $\tau^{-r}X$, contains a submodule Z with dimension vector $(1, 1)$. Therefore, there is a monomorphism $\tau^r Z \rightarrow X$, and thus, by Lemma 2.5,

$$\mu^m > \mu(M) = \mu(X[r]) \geq \mu(X) > \mu(\tau^r Z) > \{1, 2, 3\},$$

which is a contradiction. Since there is a short exact sequence

$$0 \rightarrow X[r] \rightarrow X[r + 1] \rightarrow \tau^{-r}X \rightarrow 0$$

and $\text{Hom}(Y, \tau^{-r}X) = 0$, the inclusion $Y \rightarrow X[r + 1]$ factors through $X[r]$. Therefore, there is a monomorphism $Y \rightarrow X[r]$. It follows that

$$\mu(X[r]) \geq \mu(Y) = \mu^t \geq \mu^m > \mu(M).$$

This contradiction implies that μ^m has no direct predecessor for each $m \geq 1$. In particular, $\text{ndp}(Q) = \infty$. \square

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